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Some reformulations and extensions of the theory of rhythmic canons

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ABSTRACT

The algebraic theory of periodic rhythmic canons was developed by the author of the present paper in connection with the study of a special class of rhythmic canons, nowadays referred as "Vuza canons". A basic concept in the theory is the outer rhythm attached to a canon. In the original paper, for any given canon an auxiliary canon was constructed and the outer rhythm was expressed as an object attached to the latter canon. In the present paper we show that the outer rhythm can be expressed directly in terms of the given canon, without the need of an auxiliary construction. Strongly related to the outer rhythm is the canon category, which is a numerical measure of the periodic symmetry of the outer rhythm relative to the canon inner rhythm. We give a new definition of the category in terms of the stability groups associated to a canon. Many interesting results due to various authors have been obtained for Vuza canons, which by their definition must have maximal category. We show here that interesting facts can also be said about canons whose category is not maximal. We describe partitions of a canon into subcanons of minimal and maximal category and we discuss the relation between outer rhythm, category and a class of maps that can be regarded as natural morphisms between canons.

Introduction

The author developed the algebraic theory of periodic rhythm (Vuza, 1985, 1989) that subsequently served as the foundation of a theory of periodic rhythmic canons exposed in Vuza (1991, 1192a, 1992b, 1993); see also Vuza (1995) for a less technical presentation. In the latter theory the author studied especially the properties of a certain class of canons that he called regular complementary of maximal category. These canons attracted much interest from music theorists as well as from musicians. They were called Vuza canons by those who carried further the research on this subject. For a review of some of these developments, see for instance (Agon and Andreatta, 2011; Andreatta, 2011).

One concept that is essential for the theory of Vuza canons is the category of a canon. It was introduced in Vuza (1991, 1192a, 1992b, 1993) as an extension of a concept employed in Grigor'ev and Muller (1961), where canons on two voices were classified as category one and category two. The category of a canon as defined in Vuza (1991, 1192a, 1992b, 1993) is an integer divisor of the number N of voices in a canon. At extremes we have maximal category (equal to N) and minimal category (equal to one). When N is prime these are the single possibilities. When N is not prime there may be intermediate situations.

As their original name implied, all Vuza canons have maximal category. It is the author's belief that interesting facts can be found outside of the realm of maximal category. It is the purpose of this paper to pinpoint at some of those facts. Its organization is as follows.

The following section is a brief presentation of the essentials of the theory of rhythms and canons, with the purpose of establishing the framework for the next sections.

The concept of canon category as defined in Vuza (1991, 1192a, 1992b, 1993) relied on the concept of outer rhythm of a canon introduced in Vuza (1991, 1192a, 1992b, 1993) with the aid of an auxiliary construction based on regular rhythms. Even if the end result was proved to not depend on the particular way of making this construction, the regular rhythms employed herein neither were part of the canon under study, nor their choice was unique. On following, we attempt to define the outer rhythm and category in an "intrinsic" way, that is, by relying only on characteristics that are unambiguously present in the canon under study. The fact that the definitions are given in two separate sections is to emphasize that in their new form, each of them can be given independently from the other.

There is a connection between category and outer rhythms, which is below discussed. In particular it is proved there that the

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canons of minimal category are precisely the canons for which the outer rhythms are regular.

Also, we study translation preserving maps. They provide a class of "natural morphisms" between canons. In particular one proves that maximal category can be characterized in terms of those maps: a canon **C** has maximal category if and only if the identity is the only translation preserving map on **C**. Moreover, these maps play a role in a procedure for computing the category. Additionally, one proves that the new definitions of outer rhythm and category agree with the original ones.

The last three sections are devoted to a theory of canon partitions, introducing the general concepts, discussing partitions into canons of minimal category, presenting the essentials subcanons and describing their role in constructing partitions into subcanons of maximal category.

We hope that the present paper succeeded to reveal some new aspects of the theory of rhythmic canons that would be interesting for music theorists as well as for musicians. For instance, musicians might find an interest in Proposition 6.7 and the discussion preceding it, that show how translation preserving maps can be used to "synchronize" two canons. As concerns the theory of partitions, musicians might for instance use it for assigning different melodic lines to different voices (or groups of voices) in a canon.

Musicians might find interesting to learn that the same canon can be partitioned into subcanons of minimal category and into subcanons of maximal category. Thus, the two extremes may coexist under the arch of the same canon.

A short introduction to the algebraic theory of rhythmic canons

Let $\mathbb{Q}$ denote the rationals and $\mathbb{Z}$ the integers.

For any $r \in \mathbb{Q}$ let $r\mathbb{Z}$ denote the set $\{rn \mid n \in \mathbb{Z}\}$.

We write "iff" as an abbreviation for "if and only if".

Definition 2.1. (Vuza, 1985; Vuza, 1989) *A periodic rhythm is a subset R of the rational numbers $\mathbb{Q}$ that satisfies the following conditions:*

i) *There is $t \in \mathbb{Q}$, $t > 0$ such that $\{t + r \mid r \in R\} = R$.*

ii) *R is locally finite, that is, the sets $R \cap (a, b)$ are finite for any $a \in \mathbb{Q}$, $b \in \mathbb{Q}$, $a < b$.*

The smallest positive rational number that satisfies i) is called the period of R and is denoted by $\mathrm{Per}(R)$.

In this definition, $\mathbb{Q}$ is considered as the axis of time and the elements of the subset R are interpreted as time markers of the events in the musical speech delivered by a voice or an instrument (by event we mean, for example, the attack of a note, the beat of a percussion instrument or the beginning of a silence).

The rhythms in the examples of this paper are subsets of $\mathbb{Z}$. This is for convenience only. For real musical practice the usage of $\mathbb{Q}$ presents the advantages of a direct correspondence between musical notation and numbers (such as a quarter corresponds to 1/4) and of the possibility of having an unified formal representation of all kinds of rhythmic divisions, both regular (such as quarters and eights) and irregular (such as triplets and quintuplets).

There is an action of the additive group of $\mathbb{Q}$ on the set of periodic rhythms defined by $t + R = \{t + r \mid r \in R\}$ for any $t \in \mathbb{Q}$ and for any periodic rhythm R.

We say that a $t \in \mathbb{Q}$ stabilizes a rhythm R if $t + R = R$. The stability group of a rhythm R, that is the set $\{t \mid t \in \mathbb{Q}, t + R = R\}$, is seen to be equal to $\mathrm{Per}(R)\mathbb{Z}$**.** The stability group acts on R via $(t, r) \to t + r$ and defines thus a decomposition of R into orbits. The number of such orbits equals the number of elements of any set $R \cap [a, a + \mathrm{Per}(R))$ (which is finite since R is locally finite) and is called the number of beats per period of R.

Definition 2.2. (Vuza, 1985, 1989) *A regular rhythm is a periodic rhythm for which the number of beats per period equals one. Equivalently, it is a rhythm of the form $t + r\mathbb{Z}$ with $t \in \mathbb{Q}$, $r \in \mathbb{Q}$, $r > 0$.*

Definition 2.3. (Vuza, 1985, 1989) *Two periodic rhythms R and S are called equivalent if they belong to the same orbit with respect to the action of $\mathbb{Q}$, that is, if there exists $t \in \mathbb{Q}$ such that $t + R = S$.*

Any equivalence class for the above relation is called a rhythmic class.

If $\mathscr{R}$ is a rhythmic class we define $\mathrm{Per}(\mathscr{R})$ *as the period of any rhythm belonging to $\mathscr{R}$.*

Definition 2.4. (Vuza, 1991, 1192a, 1992b, 1993, 1995) *A periodic rhythmic canon is a finite set*

$$\mathbf{C} = \{R_1, \ldots, R_N\} \tag{2.1}$$

of equivalent periodic rhythms. The common class of these rhythms is called the inner class of **C**. *The common period of these rhythms is called the period of* **C** *and is denoted by* $\mathrm{Per}(\mathbf{C})$. *The number N of rhythms in* **C** *is called the number of voices.*

Since all rhythms and canons to be considered below will be periodic, we shall omit the adjective "periodic".

The canon (2.1) models an ensemble of N voices in which each voice repetitively executes (in a loop) a rhythmic pattern that is the same for all voices, except for a delay between any pair of voices constant through the duration of the canon.

A canon as defined here is an unordered set; thus $\{R, S\}$ and $\{S, R\}$ represent the same canon. Equality of canons is to be understood in the set-theoretic sense: two canons $\mathbf{C}_1$ and $\mathbf{C}_2$ are equal if for every $R \in \mathbf{C}_1$ we have $R \in \mathbf{C}_2$ and for every $S \in \mathbf{C}_2$ we have $S \in \mathbf{C}_1$.

It should be emphasized that the presented model refers only to the rhythmic aspect, ignoring the melodic aspect of the canon.

Canons will be denoted by bold characters, possibly endowed with subscripts.

Definition 2.5. (Vuza, 1991, 1192a, 1992b, 1993, 1995) Let **C** *be a canon as in* (2.1). *The resultant of* **C** *is the rhythm* $\mathrm{Res}(\mathbf{C})$ *equal to the set-theoretic union $\bigcup_{i=1}^{N} R_i$ of the rhythms in* **C**. *The resultant class of* **C**, *denoted by* $\mathrm{RES}(\mathbf{C})$, *is the rhythmic class of* $\mathrm{Res}(\mathbf{C})$.

The next four definitions are reproduced from Vuza (1991, 1192a, 1992b, 1993) where the concepts of outer rhythm and of category of a canon were introduced with the aid of an auxiliary construction called "meter of order k". In the next two sections we shall give new definitions of the outer rhythm and of the category that rely solely on intrinsic properties of a canon and do not make use of any auxiliary construction, while in Section 6 we shall prove that the new definitions agree with the original ones.

Definition 2.6. *Let* $\mathbf{C} = \{R_1, \ldots, R_N\}$ *be a canon on* N *voices. A meter of order* $k \geq 1$ on $\mathbf{C}$ *is a canon* $\mathbf{M} = \{S_1, \ldots, S_N\}$ *formed of regular rhythms and satisfying the conditions below:*

i) $\mathrm{Per}(\mathbf{M}) = k\,\mathrm{Per}(\mathbf{C})$;
ii) *For any couple* $i, j \in \{1, \ldots, N\}$, *there is* $t_{ij} \in \mathbb{Q}$ *such that* $R_i = t_{ij} + R_j$ *and* $S_i = t_{ij} + S_j$.

In the original definition from Vuza (1991, 1192a, 1992b, 1993) the condition $S_i \subset R_i$ for $1 \leq i \leq N$ was also imposed. It turns out this is not needed for what follows.

Definition 2.7. *A metric class of order k associated to a canon* $\mathbf{C}$ *is the resultant class of a meter of order k on* $\mathbf{C}$.

For a given $k > 1$ several metric classes of order k may be associated to a canon $\mathbf{C}$. However, there is only one metric class of order 1 associated to $\mathbf{C}$.

Definition 2.8. *The outer rhythmic class of a canon* $\mathbf{C}$ *is the metric class of order* 1 *associated with* $\mathbf{C}$. *Any rhythm belonging to that rhythmic class is called an outer rhythm for* $\mathbf{C}$.

Definition 2.9. *The category of the canon* $\mathbf{C}$ *is the integer obtained by dividing the number of voices N by the integer* $\mathrm{Per}(\mathbf{C})/\mathrm{Per}(S)$, *where S is an outer rhythm for* $\mathbf{C}$. *The category of a canon is called maximal (respectively minimal) if it is equal to the number of voices (respectively equal to one).*

The fact that $\mathrm{Per}(\mathbf{C})/\mathrm{Per}(S)$ is an integer that divides the number of voices was proved in Vuza (1991, 1192a, 1992b, 1993). It will be proved again with other arguments in the following.

Definition 2.10. (Vuza, 1991, 1192a, 1992b, 1993, 1995) *A rhythmic canon* $\mathbf{C}$ *as in* (2.1) *is called regular complementary if the following conditions are satisfied:*

i) $R_i \cap R_j = \emptyset$ *for any* $i \neq j$;
ii) $\mathrm{Res}(\mathbf{C})$ *is a regular rhythm.*

The simplest regular complementary canons are of minimal category.

Regular complementary canons of maximal category were studied in Vuza (1991, 1192a, 1992b, 1993) and are nowadays referred as "Vuza canons". Their construction is rather complex. For instance a theorem proved in Vuza (1991, 1192a, 1992b, 1993) states that a Vuza canon needs at least six voices.

However, in this paper we shall not focus on Vuza canons but on other classes of canons and canon constructions that may be interesting for the musician.

An intrinsic definition of outer rhythms

In the previous section we have described the action of $\mathbb{Q}$ on the set of rhythms. In addition to that we may introduce an action of $\mathbb{Q}$ on the set of canons: if $\mathbf{C}$ is a canon as in (2.1) we define

$$t + \mathbf{C} = \{t + R_1, \ldots, t + R_N\}. \tag{3.1}$$

We say that a $t \in \mathbb{Q}$ stabilizes a canon $\mathbf{C}$ if $t + \mathbf{C} = \mathbf{C}$. According to what was said in Section 2 about the equality of canons, the fact that t stabilizes $\mathbf{C}$ means that the rhythms in the right hand of (3.1) are a permutation of the rhythms in the right hand of (2.1).

Definition 3.1. *For any periodic rhythm R and any canon* $\mathbf{C}$ *let* $\mathrm{Out}(R, \mathbf{C})$ *be the set* $\{t \mid t \in \mathbb{Q}, t + R \in \mathbf{C}\}$.

Proposition 3.2.

i) $\mathrm{Out}(R, \mathbf{C}) \neq \emptyset$ *iff R is equivalent to the rhythms of* $\mathbf{C}$.
ii) *For any* $t \in \mathbb{Q}$ *we have* $\mathrm{Out}(t + R, \mathbf{C}) = -t + \mathrm{Out}(R, \mathbf{C}) = \mathrm{Out}(R, -t + \mathbf{C})$.
iii) *If* $\emptyset \neq \mathrm{Out}(R, \mathbf{C}_1) \subset \mathrm{Out}(R, \mathbf{C}_2)$ *then* $\mathbf{C}_1 \subset \mathbf{C}_2$.
iv) *If* $0 \in R$ *then* $\mathrm{Out}(R, \mathbf{C}) \subset \mathrm{Res}(\mathbf{C})$.
v) *Assuming* $\mathrm{Out}(R, \mathbf{C}) \neq \emptyset$, a $t \in \mathbb{Q}$ *stabilizes* $\mathrm{Out}(R, \mathbf{C})$ *iff it stabilizes* $\mathbf{C}$.
vi) $\mathrm{Out}(R, \mathbf{C})$ *is a periodic rhythm whose period divides* $\mathrm{Per}(\mathbf{C})$.

Proof: i) and ii) follow immediately from definitions.

For iii), since $\mathrm{Out}(R, \mathbf{C}_1) \neq \emptyset$, for any $R_1 \in \mathbf{C}_1$ there is t such that $t + R = R_1$. This implies $t \in \mathrm{Out}(R, \mathbf{C}_1)$ and hence $t \in \mathrm{Out}(R, \mathbf{C}_2)$, meaning that $R_1 = t + R \in \mathbf{C}_2$.

For iv), if $t \in \mathrm{Out}(R, \mathbf{C})$ then $t + R \in \mathbf{C}$ and also $t \in t + R$ as $0 \in R$, therefore t is contained in a rhythm of $\mathbf{C}$ and so $t \in \mathrm{Res}(\mathbf{C})$.

v) is a consequence of iii) and of the second equality in ii).

Since $\mathrm{Per}(\mathbf{C})$ stabilizes $\mathbf{C}$, it follows from v) that it stabilizes $\mathrm{Out}(R, \mathbf{C})$ which means that $\mathrm{Out}(R, \mathbf{C})$ is a periodic subset whose period divides $\mathrm{Per}(\mathbf{C})$. When $0 \in R$, iv) shows that $\mathrm{Out}(R, \mathbf{C})$ is locally finite since it is included in a finite union of locally finite subsets. In the general case, let t_0 be any element of R. Since $0 \in -t_0 + R$ it follows that $\mathrm{Out}(-t_0 + R, \mathbf{C})$ is locally finite, hence $\mathrm{Out}(R, \mathbf{C})$ is also locally finite because of the first equality in ii). □

It follows from Proposition 3.2 that the rhythmic class of $\mathrm{Out}(R, \mathbf{C})$ does not depend on the choice of R as long as $\mathrm{Out}(R, \mathbf{C}) \neq \emptyset$.

Definition 3.3. *Any rhythm* $\mathrm{Out}(R, \mathbf{C}) \neq \emptyset$ *is called an outer rhythm of* $\mathbf{C}$. *The outer class* $\mathrm{OUT}(\mathbf{C})$ *is the rhythmic class of the outer rhythms of* $\mathbf{C}$.

All outer rhythms considered in the following will be assumed non void.

Proposition 3.4. *Let the canon* $\mathbf{C}$ *consist of regular rhythms and let* $0 \in R$. *Then* $\mathrm{Out}(R, \mathbf{C}) = \mathrm{Res}(\mathbf{C})$.

Proof: by Proposition 3.2 iv) we already know that $\mathrm{Out}(R, \mathbf{C}) \subset \mathrm{Res}(\mathbf{C})$. To prove the converse inclusion, let s belong to some rhythm of $\mathbf{C}$. The latter is necessary of the form $t + R$, hence $s = t + r$ for some $r \in R$. Since R is regular and $0 \in R$ we have $r + R = R$. Therefore $s + R = t + r + R = t + R$ and $t + R \in \mathbf{C}$, meaning that $s \in \mathrm{Out}(R, \mathbf{C})$. □

An intrinsic definition for the category of a canon

Let $\mathbf{C}$ be any canon on N voices as in (2.1).

Let $O(\mathbf{C})$ be the stability group of $\mathbf{C}$ for the action defined in Section 3, that is, the set of $t \in \mathbb{Q}$ for which $t + \mathbf{C} = \mathbf{C}$.

Let $I(\mathbf{C})$ be the stability group of any R_i in $\mathbf{C}$, that is, the set of $t \in \mathbb{Q}$ for which $t + R_i = R_i$; from the definition of a canon, $I(\mathbf{C})$ does not depend on the choice of R_i in $\mathbf{C}$.

$I(\mathbf{C})$ and $O(\mathbf{C})$ are locally finite subgroups of $\mathbb{Q}$ and $I(\mathbf{C}) \subset O(\mathbf{C})$, therefore the quotient group $O(\mathbf{C})/I(\mathbf{C})$ is finite.

The group $O(\mathbf{C})$ acts on $\mathbf{C}$ via the restriction of the action of $\mathbb{Q}$.

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Proposition 4.1. *The number N of voices in the canon* $\mathbf{C}$ *equals the number of elements of the group* $O(\mathbf{C})/I(\mathbf{C})$ *times the number of orbits of the action of* $O(\mathbf{C})$ *on* $\mathbf{C}$.

Proof: according to a well-known result from algebra, whenever a commutative group G acts on a set, there is a canonical bijection between the orbit of any element in the set and the quotient of G by the stability group of that element.

When applying the above result to the action of $O(\mathbf{C})$ on $\mathbf{C}$, we first observe that $I(\mathbf{C})$ is the common stability group for any element in $\mathbf{C}$. Therefore, there exist canonical bijections between any orbit of the named action and the finite group $O(\mathbf{C})/I(\mathbf{C})$. It follows that all orbits have the same number of elements, equal to the number of elements of the group $O(\mathbf{C})/I(\mathbf{C})$. Since the orbits realize a partition of $\mathbf{C}$, the result follows. □

Definition 4.2. *The category of a canon* $\mathbf{C}$ *on N voices is the quotient of N by the number of elements in* $O(\mathbf{C})/I(\mathbf{C})$.

According to Proposition 4.1, the category is an integer that divides N. Minimal category equals one, maximal category equals N. The category of a canon with a prime number of voices can be either maximal or minimal.

From the above discussion we have immediately the following result.

Proposition 4.3.

i) *The category of* $\mathbf{C}$ *equals the number of orbits of the action of* $O(\mathbf{C})$ *on* $\mathbf{C}$.

ii) $\mathbf{C}$ *has minimal category iff* $O(\mathbf{C})$ *acts transitively on* $\mathbf{C}$.

iii) $\mathbf{C}$ *has maximal category iff* $O(\mathbf{C}) = I(\mathbf{C})$.

The relation between category and outer rhythms

Proposition 5.1. *Let* $\mathbf{C}$ *be a canon on N voices and let S be an outer rhythm of* $\mathbf{C}$. *The category of* $\mathbf{C}$ *equals N divided by the integer* $\mathrm{Per}(\mathbf{C})/\mathrm{Per}(S)$.

Proof: By definition the category equals N divided by the number of elements of the group $O(\mathbf{C})/I(\mathbf{C})$. We have $I(\mathbf{C}) = \mathrm{Per}(\mathbf{C})\mathbb{Z}$ by the definition of $\mathrm{Per}(\mathbf{C})$. As concerns $O(\mathbf{C})$, it is not only the stability group of $\mathbf{C}$ but also the stability group of S, as shown by Proposition 3.2 v). Therefore we have $O(\mathbf{C}) = \mathrm{Per}(S)\mathbb{Z}$. Since $I(\mathbf{C}) \subset O(\mathbf{C})$, $\mathrm{Per}(\mathbf{C})/\mathrm{Per}(S)$ is an integer. It is easily proved that $\mathrm{Per}(S)\mathbb{Z}/\mathrm{Per}(\mathbf{C})\mathbb{Z}$ is isomorphic with the group of integers modulo $\mathrm{Per}(\mathbf{C})/\mathrm{Per}(S)$. Therefore, the number of elements in $O(\mathbf{C})/I(\mathbf{C})$ equals $\mathrm{Per}(\mathbf{C})/\mathrm{Per}(S)$. □

Proposition 5.2. *The category of a canon equals the number of beats per period of any of its outer rhythms.*

Proof: let $S = \mathrm{Out}(R, \mathbf{C})$ be an outer rhythm of the canon $\mathbf{C}$. By Proposition 4.3 i), the category of $\mathbf{C}$ equals the number of orbits of the action of $O(\mathbf{C})$ on $\mathbf{C}$. By its definition, the number of beats per period of S equals the number of orbits for the action on S of the stability group of S. By Proposition 3.2 v), the latter group equals $O(\mathbf{C})$.

Every action of a group on a set defines a partition of that set into orbits and every partition of a set defines an equivalence relation on that set.

Let us then consider the equivalence relations on S and on $\mathbf{C}$ defined by the actions of $O(\mathbf{C})$ and let $U : S \to \mathbf{C}$ be the map defined by $U(t) = t + R$ for any $t \in S = \mathrm{Out}(R, \mathbf{C})$. We prove that U is compatible in both directions with the named relations, that is, s_1 and s_2 are equivalent iff $U(s_1)$ and $U(s_2)$ are equivalent.

Let $s_1,\ s_2 \in S$ be equivalent. By definition this means there is $t \in O(\mathbf{C})$ such that $s_2 = t + s_1$. Then $U(s_2) = s_2 + R = t + s_1 + R = t + U(s_1)$, showing that $U(s_1)$ and $U(s_2)$ are equivalent.

Let $s_1,\ s_2 \in S$ be such that $U(s_1)$ and $U(s_2)$ are equivalent. By definition this means there is $t \in O(\mathbf{C})$ such that $U(s_2) = t + U(s_1)$. By the definition of U this means that $s_2 + R = t + s_1 + R$. It follows that $t + s_1 - s_2$ belongs to the stability group of R, which equals $I(\mathbf{C})$ since R must be in the same rhythmic class as the rhythms in $\mathbf{C}$. Since $I(\mathbf{C}) \subset O(\mathbf{C})$ we have $t + s_1 - s_2 \in O(\mathbf{C})$ and since $t \in O(\mathbf{C})$ we also have $s_1 - s_2 \in O(\mathbf{C})$ showing that s_1 and s_2 are equivalent.

Let $\hat{U}$ be the map between the set of orbits of S and the set of orbits of $\mathbf{C}$ that maps the orbit of s into the orbit of $U(s)$. By what was proved above, $\hat{U}$ is correctly defined and injective. Since U is surjective, $\hat{U}$ is also surjective. Therefore $\hat{U}$ establishes a bijection between the set of orbits of S and the set of orbits of $\mathbf{C}$, showing in particular that these sets have the same number of elements. □

Proposition 5.3. *A canon has minimal category iff its outer rhythms are regular.*

Proof: it follows from Proposition 5.2 taking into account that regular rhythms are precisely the rhythms with one beat per period. □

Translation preserving maps

Definition 6.1. *Let* $\mathbf{C}_1$ *and* $\mathbf{C}_2$ *be canons. A map* $\pi : \mathbf{C}_1 \to \mathbf{C}_2$ *is called translation preserving if for any* $t \in \mathbb{Q}$ *and* $R, S \in \mathbf{C}_1$, *the relation* $t + R = S$ *implies* $t + \pi(R) = \pi(S)$.

The order relation $\leq$ between rhythmic classes was defined in Vuza (1985): given rhythmic classes $\mathscr{R}_1$ and $\mathscr{R}_2$, we write $\mathscr{R}_1 \leq \mathscr{R}_2$ if there are rhythms $R_1 \in \mathscr{R}_1$ and $R_2 \in \mathscr{R}_2$ such that $R_1 \subset R_2$ in the set-theoretic sense.

The antisymmetry of the above defined relation may be established as a consequence of the local finitude of rhythms.

Proposition 6.2. *For any canons* $\mathbf{C}_1$ *and* $\mathbf{C}_2$ *the following are equivalent.*

i) *There exists a translation preserving map* $\pi : \mathbf{C}_1 \to \mathbf{C}_2$.

ii) $\mathrm{Per}(\mathbf{C}_2)$ *divides* $\mathrm{Per}(\mathbf{C}_1)$ *and* $\mathrm{OUT}(\mathbf{C}_1) \leq \mathrm{OUT}(\mathbf{C}_2)$.

Proof: suppose i) is satisfied. Let R be any rhythm in $\mathbf{C}_1$. Since $\mathrm{Per}(\mathbf{C}_1) + R = R$ it follows from the definition of a translation preserving map that $\mathrm{Per}(\mathbf{C}_1) + \pi(R) = \pi(R)$. Hence $\mathrm{Per}(\mathbf{C}_1)$ stabilizes $\pi(R) \in \mathbf{C}_2$ which implies that $\mathrm{Per}(\mathbf{C}_2)$ must divide $\mathrm{Per}(\mathbf{C}_1)$. Let now $t \in \mathrm{Out}(R, \mathbf{C}_1)$. Since $R \in \mathbf{C}_1$ and $t + R \in \mathbf{C}_1$, again by the definition of a translation preserving map we obtain $t + \pi(R) = \pi(t + R) \in \mathbf{C}_2$ which implies $t \in \mathrm{Out}(\pi(R), \mathbf{C}_2)$. Therefore $\mathrm{Out}(R, \mathbf{C}_1) \subset \mathrm{Out}(\pi(R), \mathbf{C}_2)$; as the former is a rhythm of class $\mathrm{OUT}(\mathbf{C}_1)$ and the latter is of class $\mathrm{OUT}(\mathbf{C}_2)$, this proves $\mathrm{OUT}(\mathbf{C}_1) \leq \mathrm{OUT}(\mathbf{C}_2)$.

Suppose ii) is satisfied. Let R_1 be any rhythm in $\mathbf{C}_1$ and let S_2 be any rhythm in $\mathbf{C}_2$. By the definition of the order of rhythmic classes there is t such that $\mathrm{Out}(R_1, \mathbf{C}_1) \subset t + \mathrm{Out}(S_2, \mathbf{C}_2)$. By Proposition 3.2 ii) we have $t + \mathrm{Out}(S_2, \mathbf{C}_2) = \mathrm{Out}(-t + S_2, \mathbf{C}_2)$. Setting $R_2 = -t + S_2$ we have by the above $\mathrm{Out}(R_1, \mathbf{C}_1) \subset \mathrm{Out}(R_2, \mathbf{C}_2)$. To construct π, take any $R \in \mathbf{C}_1$. There is t such that $R = t + R_1$. The latter equality implies that $t \in \mathrm{Out}(R_1, \mathbf{C}_1)$ so we will also have $t \in \mathrm{Out}(R_2, \mathbf{C}_2)$, meaning that $t + R_2 \in \mathbf{C}_2$.

Define $\pi(R) = t + R_2$. We must prove that the definition does not depend on the choice of t. Let s be another rational that satisfies $R = s + R_1$. It follows that $s - t$ stabilizes R_1, hence is a multiple of Per($\mathbf{C}_1$). Since Per($\mathbf{C}_2$) divides Per($\mathbf{C}_1$), we have that $s - t$ also stabilizes R_2 which implies $s + R_2 = t + R_2$ showing that the definition is correct. □

Proposition 6.3. If Per($\mathbf{C}_1$) = Per($\mathbf{C}_2$) *and* $\pi : \mathbf{C}_1 \to \mathbf{C}_2$ *is translation preserving then the following are true.*

i) *π is injective.*

ii) *If π is bijective then π^{-1} is translation preserving.*

Proof: for i), let $R, S \in \mathbf{C}_1$ be such that $\pi(R) = \pi(S)$. There is t such that $t + R = S$. Since π is translation preserving this implies $t + \pi(R) = \pi(S)$. Together with $\pi(R) = \pi(S)$ this shows that t stabilizes $\pi(R) \in \mathbf{C}_2$, hence Per($\mathbf{C}_2$) divides t. Since Per($\mathbf{C}_1$) = Per($\mathbf{C}_2$) we obtain that t also stabilizes R, hence $R = t + R = S$ which completes the proof.

For ii), let $R, S \in \mathbf{C}_2$ be such that $t + R = S$. We have to prove that $t + \pi^{-1}(R) = \pi^{-1}(S)$. Since $\pi^{-1}(R)$ and $\pi^{-1}(S)$ belong to $\mathbf{C}_1$ there is u such that $u + \pi^{-1}(R) = \pi^{-1}(S)$, which implies $u + R = S$ as π is translation preserving. Therefore $t - u$ stabilizes R and therefore is a multiple of Per($\mathbf{C}_2$) = Per($\mathbf{C}_1$). Hence $t - u$ also stabilizes $\pi^{-1}(R)$ from which we get $t + \pi^{-1}(R) = u + \pi^{-1}(R) = \pi^{-1}(S)$. □

Proposition 6.4. *Any translation preserving map* $\pi : \mathbf{C} \to \mathbf{C}$ *is bijective, its inverse is translation preserving and there is* $t \in O(\mathbf{C})$ *such that* $\pi(R) = t + R$ *for any* $R \in \mathbf{C}$.

Conversely, for any $t \in O(\mathbf{C})$, *the map* $\pi : \mathbf{C} \to \mathbf{C}$ *defined by* $\pi(R) = t + R$ *is translation preserving.*

Proof: clearly if $t \in O(\mathbf{C})$ then the map $\pi : \mathbf{C} \to \mathbf{C}$ given by $\pi(R) = t + R$ is well defined and translation preserving.

Conversely, let $\pi : \mathbf{C} \to \mathbf{C}$ be any translation preserving map and let R_0 be some rhythm in $\mathbf{C}$. Since R_0 and $\pi(R_0)$ both belong to $\mathbf{C}$, there is t_0 such that $t_0 + R_0 = \pi(R_0)$. Let now R be any other rhythm in $\mathbf{C}$. There is t such that $R = t + R_0$. Since π is translation preserving we obtain $\pi(R) = t + \pi(R_0) = t + t_0 + R_0 = t_0 + t + R_0 = t_0 + R$. Therefore $t_0 + \mathbf{C} = \mathbf{C}$ and $\pi(R) = t_0 + R$. Since the inverse of π is given by $\pi^{-1}(R) = -t_0 + R$, it is also translation preserving. □

Proposition 6.5. *For any canon* $\mathbf{C}$ *the following are equivalent.*

i) *The identity is the only translation preserving map* $\pi : \mathbf{C} \to \mathbf{C}$.

ii) *The category* of $\mathbf{C}$ *is maximal.*

Proof: we know from Proposition 4.3 iii) that the category of $\mathbf{C}$ is maximal iff every stabilizer of $\mathbf{C}$ is a multiple of Per($\mathbf{C}$), which according to Proposition 6.4 means that the identity is the only translation preserving map on $\mathbf{C}$. □

Proposition 6.6. *For any canons* $\mathbf{C}_1$ *and* $\mathbf{C}_2$ *the following are equivalent.*

i) *There exists a bijective translation preserving map* $\pi : \mathbf{C}_1 \to \mathbf{C}_2$ *whose inverse is also translation preserving.*

ii) Per($\mathbf{C}_1$) = Per($\mathbf{C}_2$) *and there exists a bijective translation preserving map* $\pi : \mathbf{C}_1 \to \mathbf{C}_2$.

iii) $\mathbf{C}_1$ *and* $\mathbf{C}_2$ *have the same number of voices, the same period and* Out($\mathbf{C}_1$) $\leq$ Out($\mathbf{C}_2$).

iv) $\mathbf{C}_1$ *and* $\mathbf{C}_2$ *have the same number of voices, the same period and the same outer class.*

Proof: by Proposition 6.2 applied to π and to π^{-1}, i) ⇒ ii) and i) ⇒ iv). By Proposition 6.3, ii) ⇒ i). Clearly iv) ⇒ iii).

It remains to prove iii) ⇒ i). By Proposition 6.2 there exists a translation preserving map $\pi : \mathbf{C}_1 \to \mathbf{C}_2$. Since Per($\mathbf{C}_1$) = Per($\mathbf{C}_2$), Proposition 6.3 implies that π is injective, therefore bijective as it acts between sets with equal numbers of elements. By Proposition 6.3, this implies that π^{-1} is translation preserving. □

We may now consider the relation between the new and the original definitions of outer rhythm and category.

Within the algebraic framework developed here, it is seen that giving a meter on a canon $\mathbf{C}$ amounts to giving a canon $\mathbf{M}$ composed of regular rhythms together with a bijective translation preserving map $\pi : \mathbf{M} \to \mathbf{C}$.

Indeed, let $\mathbf{C} = \{R_1, \ldots, R_N\}$ and let $\mathbf{M}$ and π be given. Set $S_i = \pi^{-1}(R_i)$. For every i, j there is $t_{ij} \in \mathbb{Q}$ such that $S_i = t_{ij} + S_j$. Then since π is translation preserving we also have $R_i = \pi(S_i) = t_{ij} + \pi(S_j) = t_{ij} + R_j$. On the other hand, by Proposition 6.2 the existence of π implies that Per($\mathbf{C}$) divides Per($\mathbf{M}$), therefore $\mathbf{M}$ satisfies the definition of a meter of order k = Per($\mathbf{M}$)/ Per($\mathbf{C}$) on $\mathbf{C}$.

Conversely, let $\mathbf{M} = \{S_1, \ldots, S_N\}$ be a meter of order k on $\mathbf{C}$. Define π by $\pi(S_i) = R_i$. Given i and j, let $s \in \mathbb{Q}$ be such that $s + S_i = S_j$. By the definition of a meter there is $t_{ij} \in \mathbb{Q}$ such that $t_{ij} + S_i = S_j$ and $t_{ij} + R_i = R_j$. It follows that $s - t_{ij}$ stabilizes S_i, therefore it is a multiple of Per($\mathbf{M}$), Since Per($\mathbf{M}$) is a multiple of Per($\mathbf{C}$) by the definition of a meter, it follows that $s - t_{ij}$ stabilizes also R_i. Consequently $s + \pi(S_i) = s + R_i = t_{ij} + R_i = R_j = \pi(S_j)$ which shows that π is translation preserving.

In the case of a meter of order 1, Proposition 6.6 shows that OUT($\mathbf{C}$) = OUT($\mathbf{M}$). We also know from Proposition 3.4 that in the case of a canon $\mathbf{M}$ composed of regular rhythms, OUT($\mathbf{M}$) = RES($\mathbf{M}$). Therefore OUT($\mathbf{C}$) = RES($\mathbf{M}$). We see thus that the new definition of OUT($\mathbf{C}$) agrees with the original definition of OUT($\mathbf{C}$) as the resultant class of a meter of order 1.

According to its original definition, the category of $\mathbf{C}$ was equal to the number of voices N divided by the quotient Per($\mathbf{C}$)/Per(RES($\mathbf{M}$)) where $\mathbf{M}$ was a meter of order 1 on $\mathbf{C}$. From the above discussion we have Per(RES($\mathbf{M}$)) = Per(OUT($\mathbf{C}$)). Consequently the original category equals N divided by the quotient Per($\mathbf{C}$)/Per(OUT($\mathbf{C}$)) which by Proposition 5.1 agrees with the new definition.

Translation preserving maps offer a natural way for “synchronizing” canons, by establishing a correspondence $\pi : \mathbf{C}_1 \to \mathbf{C}_2$ between two canons $\mathbf{C}_1$, $\mathbf{C}_2$ that are not necessarily built on the same inner class. Under the hypotheses specified by Proposition 6.6 iv), any such map is bijective and its inverse is also translation preserving. Consequently, if $\pi_0 : \mathbf{C}_1 \to \mathbf{C}_2$ is translation preserving, any other such map $\pi : \mathbf{C}_1 \to \mathbf{C}_2$ is of the form $\pi = \pi_0\sigma$ with $\sigma : \mathbf{C}_1 \to \mathbf{C}_1$ translation preserving. By Proposition 6.5, σ may be different of the identity map iff the category of $\mathbf{C}_1$ (equal to the category of $\mathbf{C}_2$) is not maximal. Therefore, under the conditions specified by Proposition 6.6 iv), maximal category is the necessary and sufficient condition for the unicity of the translation preserving map $\pi : \mathbf{C}_1 \to \mathbf{C}_2$.

The usefulness of “synchronizing” two canons can be also illustrated by the following construction. Let $\oplus$ be a binary law on the set of periodic rhythms that commutes with translations, in the sense that $t + (R \oplus S) = (t + R) \oplus (t + S)$; in particular $\oplus$ can be any of $\cup$, $\cap$ or \.

Proposition 6.7. *For any canons* $\mathbf{C}_1$, $\mathbf{C}_2$ *and any translation preserving map* $\pi : \mathbf{C}_1 \to \mathbf{C}_2$, *the set* $\{R \oplus \pi(R) \mid R \in \mathbf{C}_1\}$ *is a canon.*

Proof: We have to prove that all rhythms of the above set

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belong to the same rhythmic class. Let R and S be rhythms of $\mathbf{C}_1$. There is t such that $t + R = S$. Since π is translation preserving we have $t + \pi(R) = \pi(S)$. Then by the property of $\oplus$ we have $t + (R \oplus \pi(R)) = (t + R) \oplus (t + \pi(R)) = S \oplus \pi(S)$, proving thus that $R \oplus \pi(R)$ and $S \oplus \pi(S)$ belong to the same rhythmic class. □

We close with the description of a procedure for determining the category of a canon based on translation preserving maps.We start with some algebraic preliminaries.

Given a set R and an equivalence relation ρ on R, a map $f : R \to R$ is said to be compatible with ρ if $f(x_1)$ is equivalent with $f(x_2)$ whenever x_1 and x_2 are equivalent. We have already encountered a similar situation in the proof of Proposition 5.1; however, here we demand compatibility only in the forward direction. Let R_ρ be the set of equivalence classes for ρ. Starting from f we define the map $f_\rho : R_\rho \to R_\rho$ that sends the class of x into the class of $f(x)$; the compatibility of f with ρ ensures that f_ρ is well defined.

If $g : R \to R$ is another map compatible with ρ, then $(fg)_\rho = f_\rho g_\rho$. In particular this shows that $(f^k)_\rho = (f_\rho)^k$ for any integer $k \geq 1$. Likewise, if f is compatible and bijective and f^{-1} is also compatible, then f_ρ is bijective.

One situation when we may apply the above is when together with f we are given a surjective map $\varphi : R \to C$. The relation ρ is defined by calling equivalent any x_1 and x_2 for which $\varphi(x_1) = \varphi(x_2)$. Since in this case φ establishes a canonical bijection between R_ρ and C, the above construction gives a map $f_\rho : C \to C$ such that $f_\rho(\varphi(x)) = \varphi(f(x))$ for every $x \in R$.

Another situation will be when R is defined by the action of a group, as we have already seen in the preceding sections.

Consider now a rhythm R together with the action of a subgroup $G \neq \{0\}$ of the stability group of R; G is of the form $k\mathrm{Per}(R)\mathbb{Z}$ for some integer $k \geq 1$. For every $r \in R$ let the successor $\sigma(r)$ of r be the least element in R strictly greater than r and let the predecessor $\pi(r)$ of r be the greatest element in R strictly less than r. Because R is locally finite the successor and the predecessor are well defined.

Proposition 6.8. *σ and π are compatible with the equivalence defined by the action of G.*

Proof: We shall make the proof for σ, the proof for π being similar. Let r and $r + g$ be any two equivalent elements in R, with $g \in G$. Since $\sigma(r) \in R$ and g stabilizes R, we also have $\sigma(r) + g \in R$. Since $\sigma(r) > r$ we also have $\sigma(r) + g > r + g$. If there were any element s of R satisfying $r + g < s < \sigma(r) + g$, then $s - g$ would be an element of R satisfying $r < s - g < \sigma(r)$ which would contradict the definition of $\sigma(r)$. Therefore $\sigma(r + g) = \sigma(r) + g$ which shows that $\sigma(r)$ and $\sigma(r + g)$ are equivalent. □

Obviously, the maps σ and π are inverse to each other. Since both are compatible with the equivalence defined by G, it follows that the map σ_G constructed from σ on the set R_G of equivalence classes is bijective. Since R_G is a finite set we may consider the order of σ_G as an element of the finite permutation group of R_G.

Proposition 6.9. *When* $G = \mathrm{Per}(R)\mathbb{Z}$ *the order of* σ_G *equals the number of beats per period of R.*

Proof: let n be the number of beats per period of R. For any $r \in R$ there are exactly n elements in the set $R \cap [r, r + \mathrm{Per}(R))$. Consequently $\sigma^n(r) = r + \mathrm{Per}(R)$ and so $\sigma^n(r)$ and r are equivalent, implying that $(\sigma_G)^n$ is the identity map. On the other hand, if $k < n$ then $\sigma^k(r) \in [r, r + \mathrm{Per}(R))$ and hence it cannot be equivalent with r. □

Consider now a canon $\mathbf{C}$ together with its outer rhythm $R_O = \mathrm{Out}(R, \mathbf{C})$. The map that sends $r \in R_O$ into $r + R \in \mathbf{C}$ is surjective and the equivalence defined by it is precisely the equivalence defined by the action of $\mathrm{Per}(\mathbf{C})\mathbb{Z}$. Since the latter group stabilizes R_O, it follows that a map σ_C can be defined on $\mathbf{C}$ so that $\sigma(r) + R = \sigma_C(r + R)$ for every $r \in R_O$, where σ is the successor map on R_O.

Proposition 6.10. *The category of* $\mathbf{C}$ *equals the least integer* $k \geq 1$ *for which* $(\sigma_C)^k$ *is translation preserving.*

Proof: as remarked at the beginning of our discussion we have $\sigma^k(r) + R = (\sigma_C)^k(r + R)$ for every $r \in R_O$ and every $k \geq 1$. If k equals the category of $\mathbf{C}$, by Proposition 5.2 it also equals the number of beats per period of R_O. As we have seen in the proof of Proposition 6.9, in this situation $\sigma^k(r) = r + \mathrm{Per}(R_O)$. Therefore $(\sigma_C)^k(r + R) = \mathrm{Per}(R_O) + r + R$. In other words, $(\sigma_C)^k(S) = \mathrm{Per}(R_O) + S$ for any rhythm S in $\mathbf{C}$. Since $\mathrm{Per}(R_O)$ stabilizes $\mathbf{C}$ by Proposition 3.2 v), it follows from Proposition 6.4 that $(\sigma_C)^k$ is translation preserving.

Conversely, let $k \geq 1$ be such that $(\sigma_C)^k$ is translation preserving. By Proposition 6.4 there is a multiple t of $\mathrm{Per}(R_O)$ such that $(\sigma_C)^k(S) = t + S$ for every $S \in \mathbf{C}$. It follows that $\sigma^k(r) + R = (\sigma_C)^k(r + R) = t + r + R$, hence $\sigma^k(r) - t - r$ is a multiple of $\mathrm{Per}(\mathbf{C})$. As $\mathrm{Per}(\mathbf{C})$ is a multiple of $\mathrm{Per}(R_O)$ it follows that $\sigma^k(r) - t - r$ is a multiple of $\mathrm{Per}(R_O)$ and so is $\sigma^k(r) - r$ because t is a multiple of $\mathrm{Per}(R_O)$. Therefore $\sigma^k(r)$ and r are equivalent for the relation defined by the action of $G = \mathrm{Per}(R_O)\mathbb{Z}$ on R_O. As this is true for every $r \in R_O$ it results that $(\sigma_G)^k$ is the identity map. Hence k must be greater or equal to the order of σ_G, which by Proposition 6.9 equals the number of beats per period of R_O, which by Proposition 5.2 equals the category of $\mathbf{C}$. Hence k must be no less than the category. □

Definition 6.11. A canon $\mathbf{C}$ is in normal form if it is written as

$$\mathbf{C} = \{t_1 + R, \ldots, t_N + R\} \tag{6.1}$$

where $t_i \in [0, \mathrm{Per}(R))$ for $1 \leq i \leq N$ and $t_i < t_{i+1}$ for $1 \leq i \leq N - 1$.

Any canon may be brought to the normal form.

Proposition 6.12. *Let* $\mathbf{C}$ *be a canon in the normal form* (6.1), *let* σ *be the successor map on* $\mathrm{Out}(R, \mathbf{C})$ *and let* σ_C *be the map on* $\mathbf{C}$ *constructed from* σ*. Then:*

$$\sigma_C(t_i + R) = t_{i+1} + R \textit{ for } 1 \leq i \leq N-1,\ \sigma_C(t_N + R) = t_1 + R.$$

Proof: let $R_O = \mathrm{Out}(R, \mathbf{C})$. It is easy to establish the equality $R_O \cap [0, \mathrm{Per}(R)) = \{t_1, \ldots, t_N\}$.

Because of that relation we have $\sigma(t_i) = t_{i+1}$ for $1 \leq i \leq N-1$, hence $\sigma_C(t_i + R) = \sigma(t_i) + R = t_{i+1} + R$.

We prove that $\sigma(t_N) = \mathrm{Per}(R) + t_1$. Indeed, $\mathrm{Per}(R) + t_1 \in R_O$ since $t_1 \in R_O$ and $\mathrm{Per}(R)$ stabilizes R_O. Also, $t_N < \mathrm{Per}(R) \leq \mathrm{Per}(R) + t_1$. It remains to prove that no element of R_O exists in the interval $(t_N, \mathrm{Per}(R) + t_1)$. If such an element t existed, it must be no less that $\mathrm{Per}(R)$ because otherwise t_N would not be the greatest element of R_O in the interval $[0, \mathrm{Per}(R))$. But then $t - \mathrm{Per}(R)$ would be an element of R_O in the interval $[0, t_1)$ and so t_1 would not be the least element of R_O in the interval $[0, \mathrm{Per}(R))$.

In conclusion $\sigma_C(t_N + R) = \sigma(t_N) + R = t_1 + \mathrm{Per}(R) + R = t_1 + R$. □

The procedure for computing the category presented in the example below is justified by Propositions 6.10 and 6.12.

Consider a canon as in (6.1) but without the constraints of the normal form.

$\mathbf{C} = \{R, 3+R, 7+R, 9+R, 12+R, 13+R, 15+R, 19+R, 24+R\}$ with $\mathrm{Per}(R) = 18$.

Bring $\mathbf{C}$ to the normal form. To this purpose first replace each t_i by the uniquely determined $s_i \in [0, \mathrm{Per}(R))$ such that $s_i - t_i$ is a multiple of $\mathrm{Per}(R)$.

$\mathbf{C} = \{R, 3+R, 7+R, 9+R, 12+R, 13+R, 15+R, 1+R, 6+R\}$

Then arrange the s_i's in increasing order.

$\mathbf{C} = \{R, 1+R, 3+R, 6+R, 7+R, 9+R, 12+R, 13+R, 15+R\}$

Write the s_i's in a row and add the element $s_1 + \mathrm{Per}(R)$ at the right end of the sequence.

0, 1, 3, 6, 7, 9, 12, 13, 15, 18

Form the differences of successive elements.

1, 2, 3, 1, 2, 3, 1, 2, 3

Permute circularly one step at a time. The number of steps that take back to the start sequence is the category.

k = 0 (1, 2, 3, 1, 2, 3, 1, 2, 3)
k = 1 (2, 3, 1, 2, 3, 1, 2, 3, 1)
k = 2 (3, 1, 2, 3, 1, 2, 3, 1, 2)
k = 3 (1, 2, 3, 1, 2, 3, 1, 2, 3)

The category of $\mathbf{C}$ is 3.

Partitions of a canon

Definition 7.1. *Two canons $\mathbf{C}_1$ and $\mathbf{C}_2$ are called equivalent if they belong to the same orbit of the action of $\mathbb{Q}$, that is, if there is $t \in \mathbb{Q}$ such that $t + \mathbf{C}_1 = \mathbf{C}_2$.*

Proposition 7.2. *Equivalent canons have equal inner classes, equal outer classes and equal resultant classes.*

Proof: let $\mathbf{C}_1$ and $\mathbf{C}_2$ be canons such that $t + \mathbf{C}_1 = \mathbf{C}_2$. By Proposition 3.2 ii) we have $\mathrm{Out}(R, \mathbf{C}_2) = \mathrm{Out}(R, t + \mathbf{C}_1) = t + \mathrm{Out}(R, \mathbf{C}_1)$ for any rhythm R, therefore the outer rhythms $\mathrm{Out}(R, \mathbf{C}_1)$ and $\mathrm{Out}(R, \mathbf{C}_2)$ are equivalent.

From the definition one checks that $\mathrm{Res}(\mathbf{C}_2) = \mathrm{Res}(t + \mathbf{C}_1) = t + \mathrm{Res}(\mathbf{C}_1)$, therefore the resultant rhythms are equivalent. □

Since a canon is a finite collection $\mathbf{C}$ of equivalent periodic rhythms, it follows that every subset of $\mathbf{C}$ is itself a canon, that will be called a subcanon of $\mathbf{C}$.

Definition 7.3. *A partition of a canon $\mathbf{C}$ is a collection of mutually disjoint subcanons whose union equals $\mathbf{C}$.*

In other words, a partition of $\mathbf{C}$ is a collection $\{\mathbf{C}_i \mid 1 \leq i \leq M\}$ such that $\mathbf{C}_i \cap \mathbf{C}_j = \varnothing$ for $i \neq j$ and $\bigcup_{i=1}^{M} \mathbf{C}_i = \mathbf{C}$.

A partition of a canon $\mathbf{C}$ into equivalent subcanons is a partition formed of subcanons that are mutually equivalent as canons.

Proposition 7.4. *Let $\{\mathbf{C}_i \mid 1 \leq i \leq M\}$ be a partition of $\mathbf{C}$ into equivalent subcanons. Then the collection*

$$\{\mathrm{Res}(\mathbf{C}_i) \mid 1 \leq i \leq M\} \tag{7.1}$$

is a canon.

Proof: by Proposition 7.2 all rhythms in (7.1) are equivalent. □

Proposition 7.5. *For any partition of a regular complementary canon into equivalent subcanons, the canon constructed as in (7.1) is regular complementary.*

Proof: it follows immediately from definitions. □

Proposition 7.6. *For every partition $\{\mathbf{C}_i \mid 1 \leq i \leq M\}$ of $\mathbf{C}$ into equivalent subcanons the following are true.*

i) *$O(\mathbf{C}_i) \subset O(\mathbf{C})$ for $1 \leq i \leq M$.*

ii) *If $\mathbf{C}$ has maximal category then all $\mathbf{C}_i$ have maximal category.*

Proof:

i) let $s \in O(\mathbf{C}_i)$, meaning that $s + \mathbf{C}_i = \mathbf{C}_i$. Any other $\mathbf{C}_j$ is equivalent to $\mathbf{C}_i$, so there is $t \in \mathbb{Q}$ such that $\mathbf{C}_j = t + \mathbf{C}_i$. It follows that $s + \mathbf{C}_j = s + (t + \mathbf{C}_i) = t + (s + \mathbf{C}_i) = t + \mathbf{C}_i = \mathbf{C}_j$. We have thus proved that $s \in O(\mathbf{C}_j)$ for $1 \leq j \leq M$. Given any rhythm R in $\mathbf{C}$, by the definition of a partition there is j such that $R \in \mathbf{C}_j$. But then $s + R \in \mathbf{C}_j \subset \mathbf{C}$. We have thus proved that $s + R \in \mathbf{C}$ whenever $R \in \mathbf{C}$, meaning that $s \in O(\mathbf{C})$. As s was arbitrary in $O(\mathbf{C}_i)$, we have proved that $O(\mathbf{C}_i) \subset O(\mathbf{C})$.

ii) When $\mathbf{C}$ has maximal category we have $O(\mathbf{C}) = I(\mathbf{C})$ by Proposition 4.3 iii). Then by i) $I(\mathbf{C}) = I(\mathbf{C}_i) \subset O(\mathbf{C}_i) \subset O(\mathbf{C}) = I(\mathbf{C})$, therefore $O(\mathbf{C}_i) = I(\mathbf{C}_i)$ and $\mathbf{C}_i$ has maximal category again by Proposition 4.3 iii). □

Partitions into subcanons of minimal category

Let G be any subgroup of $O(\mathbf{C})$. Since by definition $t + R \in \mathbf{C}$ for any $t \in O(\mathbf{C})$ and $R \in \mathbf{C}$ and since $G \subset O(\mathbf{C})$, we have an action of G on $\mathbf{C}$ that defines a partition of $\mathbf{C}$ into orbits. Let $\mathbf{D}$ be any such orbit. Since G stabilizes $\mathbf{D}$ by the definition of an orbit it follows that $G \subset O(\mathbf{D})$. Also, by the definition of an orbit G acts transitively on $\mathbf{D}$ and therefore $O(\mathbf{D})$ also acts transitively on $\mathbf{D}$. By Proposition 4.3 ii), this implies that $\mathbf{D}$ has minimal category.

Proposition 8.1. *The decomposition into orbits of the canon $\mathbf{C}$ defined by the action of a subgroup of $O(\mathbf{C})$ is a partition into equivalent subcanons of minimal category.*

Proof: we have already seen that the orbits are subcanons of minimal category. Being orbits, they form of course a partition of $\mathbf{C}$. It is left to show that these subcanons are equivalent. Let $\mathbf{D}_1$ and $\mathbf{D}_2$ be two orbits. Pick rhythms $R_1 \in \mathbf{D}_1$ and $R_2 \in \mathbf{D}_2$. Since R_1 and R_2 belong to $\mathbf{C}$ they must be equivalent, hence $t + R_1 = R_2$ for some $t \in \mathbb{Q}$. Then by the definition of an orbit

$\mathbf{D}_2 = \{g + R_2 \mid g \in G\} = \{g + (t + R_1) \mid g \in G\} = \{t + (g + R_1) \mid g \in G\} = t + \{g + R_1 \mid g \in G\} = t + \mathbf{D}_1$. □

Partitions into essential subcanons

Definition 9.1. *The minimal decomposition of a canon* **C** *is the partition into orbits obtained by taking* $G = O(\mathbf{C})$ *in the construction from the preceding section.*

Definition 9.2. *An essential subcanon of the canon* **C** *is a subcanon that has in common one and only one rhythm with every orbit of the minimal decomposition.*

By acting with $O(\mathbf{C})$ on an essential subcanon we retrieve the original canon **C**. In addition, an essential subcanon is minimal with this property: by deleting a single rhythm from the subcanon it will be no longer possible to retrieve the whole canon by using the action of $O(\mathbf{C})$ only.

Proposition 9.3. *For every essential subcanon* **D** *of* **C** *the following are true.*

i) *For every* $s \in O(\mathbf{C})$, $s + \mathbf{D}$ *is an essential subcanon.*

ii) *For every* $s_1, s_2 \in O(\mathbf{C})$ *the subcanons* $s_1 + \mathbf{D}$ *and* $s_2 + \mathbf{D}$ *are either disjoint or equal, the latter situation occurring iff* $s_1 - s_2 \in I(\mathbf{C})$.

Proof:

i) for any $R \in \mathbf{C}$ the orbit of $-s + R$ has an element in common with **D**, so there is $t \in O(\mathbf{C})$ such that $t - s + R \in \mathbf{D}$. It follows that $t + R \in s + \mathbf{D}$ showing that the orbit of R has the element $t + R$ in common with $s + \mathbf{D}$. Therefore, any orbit of **C** has at least one element in common with $s + \mathbf{D}$. Suppose now that R_1 and R_2 in **D** are such that $s + R_1$ and $s + R_2$ belong to the same orbit. Hence there is $t \in O(\mathbf{C})$ such that $t + (s + R_1) = s + R_2$. But then $t + R_1 = R_2$ meaning that R_1 and R_2 belong to the same orbit, so they must coincide by the definition of **D**. Hence $s + \mathbf{D}$ has at most one element in common with each orbit.

ii) Let $s_1, s_2 \in O(\mathbf{C})$ be such that the subcanons $s_1 + \mathbf{D}$ and $s_2 + \mathbf{D}$ are not disjoint. Then there are $R_1, R_2 \in \mathbf{D}$ such that $s_1 + R_1 = s_2 + R_2$. But then R_1 and R_2 are in the same orbit, hence they must coincide as **D** has in common at most one rhythm with each orbit. It follows that $s_1 + R_1 = s_2 + R_1$ which implies that $s_1 - s_2 \in I(\mathbf{C})$ and consequently $s_1 + \mathbf{D} = s_2 + \mathbf{D}$. □

We have just proved that the set of essential subcanons of **C** is stable for the action of $O(\mathbf{C})$, so we can consider its decomposition into orbits defined by this action.

Proposition 9.4. *Every orbit of the action of* $O(\mathbf{C})$ *on the set of essential subcanons of* **C** *is a collection of subcanons that form a partition of* **C** *into equivalent subcanons.*

Proof: let **D** be an essential subcanon of **C**. Since the orbit of **D** is the set of subcanons of the form $s + \mathbf{D}$ with $s \in O(\mathbf{C})$, it is clear that all these subcanons are equivalent. By Proposition 9.3 ii) all these subcanons are mutually disjoint. It is left to show that their union equals **C**. Let R be any rhythm in **C**. Since **D** intersects every orbit of the action of $O(\mathbf{C})$ on **C**, there is $s \in O(\mathbf{C})$ such that $-s + R \in \mathbf{D}$. But then $R \in s + \mathbf{D}$ which proves the assertion. □

Proposition 9.5. *Every essential subcanon has maximal category.*

Proof: let **D** be an essential subcanon of **C** and let $s \in O(\mathbf{D})$ be given. By the definition of $O(\mathbf{D})$ we have $s + \mathbf{D} = \mathbf{D}$. By Proposition 7.6 i) we must also have $s \in O(\mathbf{C})$. But then by Proposition 9.3 ii), the equality $s + \mathbf{D} = \mathbf{D}$ can occur only if $s \in I(\mathbf{C}) = I(\mathbf{D})$. We have thus shown that $O(\mathbf{D}) \subset I(\mathbf{D})$ which proves the maximal category. □

A very simple example will help clarifying the above concepts.

Let R be any rhythm of period 8. Consider the canon

$$\mathbf{C} = \{R, 1 + R, 4 + R, 5 + R\}.$$

We have $I(\mathbf{C}) = 8\mathbb{Z}$, $O(\mathbf{C}) = 4\mathbb{Z}$ and the category of **C** equals 2.

In the following we shall write t as a shorthand for $t + R$. With this shortened notation the original canon is written

$$\mathbf{C} = \{0, 1, 4, 5\}.$$

Below we give the minimal decomposition (first line) and the list of essential subcanons (second and third lines), with equivalent subcanons placed on the same line.

$\mathbf{C} = \{0, 4] \cup \{1, 5\}$
$\{0, 1\}$ $\{4, 5\}$
$\{0, 5\}$ $\{1, 4\}$

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