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SOME FASCINATING DEVELOPMENTS IN MATHEMATICS AND MUSIC

Emmanuel Amiot

Associate researcher with LAMPS, University of Perpignan, France

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(manu.amiot@free.fr)

ABSTRACT

The strength of the bonds between music and mathematics goes without saying. This popular belief hides a subtler misconception, that this relationship involves old school mathematics: arithmetics in the Greek School (Pythagoras), diophantine approximations in tuning theory (Euler, Rameau), Fourier series for the decomposition of sound signal, and little else. However, there is much more than that and these two sciences still advance hand in hand as of today. This paper will present by way of example three musical situations involving contemporary mathematical topics: Galois theory in a rhythmic canon problem in the field of minimalist music; a graph theory question raised by Ludwig van Beethoven which had to wait almost two centuries for an answer; and a neat word theory theorem discovered in a construction originating in combinations of mystical octaves and fifths in Plato's *Timaeus*.

KEYWORDS: Mathematics, Music, Tiling, Beethoven, Graphs, Palindrom

1. ONCE UPON A TIME

A strong bond between Music and Mathematics dates back to Pythagoras and his school. The mythical master allegedly discovered so-called "natural" musical intervals by dividing the length of a string on a monochord, thus creating the first abstract model of a musical scale by a mystical correspondence with simple fractions, enforcing the philosophical credo that "All things are numbers". This philosophy dominated music theory for centuries and seems reinforced when Leibniz famously writes to Goldbach "Musica est exercitium arithmeticae". However, the whole untruncated sentence goes far beyond the Greek's reduction to integral numbers:

Musica est exercitium arithmeticae ***occultum nescientis se numerare animi***.[1]

Whatever he meant by this "unconscious activity of the soul", Leibniz certainly knew as a major actor in the field, and inventor of Calculus, that mathematics had already extended far beyond Ancient Greece's wildest dreams, with whole new fields of knowledge opening before researchers (analytical geometry, algebraic coordinates, integral and differential calculus etc.). Indeed, it was less than two decades later, in 1729, that Jean d'Alembert stated the string vibration partial differential equation, vindicating by calculus the repartition of harmonics of a metallic string (or of an air column in a wind instrument), and Euler began working on his theory of consonance and temperaments using Diophantine approximation (Euler, 1739). However, the scientific status of music (once regarded as one of the four majors arts in mediaeval *Quadrivium*) crumbled down in the XIXth century when a romantic conception of its ineffability put a damper on its relationship to mathematics and science in general. This regrettable estrangement lasted until the 1950's, when audacious young composers and theorists renewed the old bond between the two Arts in their modern forms. "La musique a rattrapé son retard sur les mathématiques", declared Iannis Xenakis ("music finally caught up with mathematics"), a major player in this development, who used relatively advanced mathematics in his composition: Markovian processes, stochastic distributions, algebraic groups, 3d-geometry (see Amiot, 2022 for references on his influence on music composition and theory). Other actors on the musical scene (for instance Babbitt, Lewin, Forte for the USA) went even further in the use of abstract mathematical notions, such as Set theory and group actions.

Nowadays, as can be seen by perusing the summaries of *JMM*[2] or the proceedings of the reunions of the SMCM[3], the relationships between maths and music encompass Category Theory, Topology, Graph Theory, Homology, Differential Calculus, Abstract Algebra, Linear Algebra, etc, it is actually difficult to find a topic in contemporary mathematics that would be unheard of in music! (Mazzola, 2016) The following examples of mathe-musical discoveries do **not** involve numbers — arithmetics — but much more abstract and refined concepts. The purpose of this choice is to put forward that music and mathematics are again developing simultaneously, in ways as far from Xenakis's knowledge than his was from Leibniz's.

2. TOM JOHNSON'S TILINGS BY AUGMENTATION

In 2001 during the *Journées d'informatique Musicale* in Bourges, American composer Tom Johnson came up with some examples of a **rhythmic canon by augmentation** together with a puzzling remark. A musical canon is made of several voices (singers, or musical instruments, or voices internal to one instrument like Bach's suites), playing the same motif but at different moments, sometimes allowing transformations (like playing the motif backwards, or upside down). In this instance, Johnson allowed his motif to be slowed down by 2 (or 4, or 8. . .) to satisfy an additional constraint: *on a given beat there should be exactly one voice playing*, no more and no less. The shortest solution appears in Fig. 1 (lines are voices, a black square mark a note played).

[1] In a letter to C. Goldbach, April 17, 1712. Quoted in Euler's *Tentamen* referenced below. It is often translated as « Music is a hidden arithmetic exercise of the soul, **which is unaware of counting** » (I underline).

[2] Journal of Mathematics and Music, Taylor & Francis.

[3] Society for Mathematics and Computation in Music. Founded in 2007 in Berlin, www.smcm-net.info.

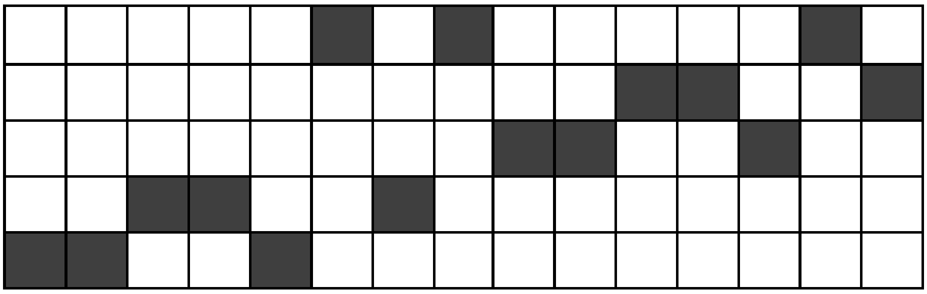

Figure 1: Johnson's problem's shortest solution. Each voice features 3 notes.

It is easy to check by trial and error that slowing down (what musicians like Bach called "augmentations") cannot be avoided. What puzzled Johnson and mathematician Andranik Tangian, who helped him find all solutions with some maximum length by a brute-force algorithm, was that *all solutions had for length a multiple of 15*. A solution with *three* ratios of augmentation and length 30=2x15 is shown on Fig. 2, where the original voice appears in blue, its augmentation by 2 in red and by 4 in black.

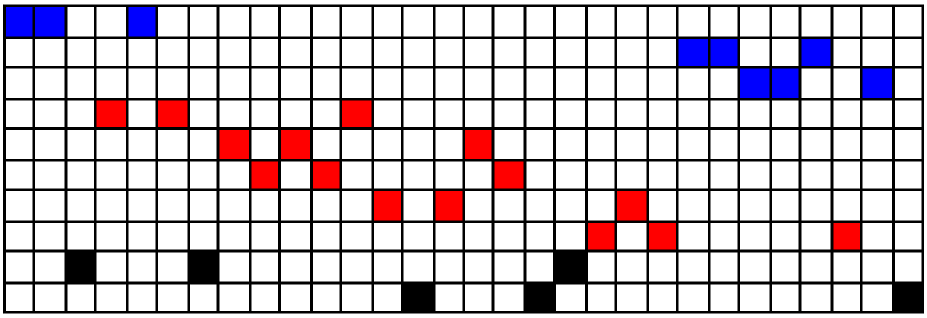

Figure 2: Another solution with length 30

For the composer, this was amusing and of little consequence: computer-made solutions provided sufficient material for his musical pieces. For a mathematician, it is an irksome question: is this a fact? Can it be proven? What of another motif? As far as I know, the only existing proof involves Galois theory of finite fields. Galois devised this theory in 1830 while solving the open problem of finding formulas for solutions of polynomials equations (the answer is "no way"). His breakthrough consisted in studying symmetries between roots of polynomials – in more general terms, trying to find unavoidable relationships between solutions of a same problem: for instance, a multiple root happens when several solutions happen to be *identical*.

Johnson's problem can be formalized as finding several combinations of transforms of the polynomial $1+X+X^4$ which would add up to a sum of consecutive powers. For the shortest solution in Fig. 1 this can be expressed as

$$(1+X+X^4)+X^2(1+X+X^4)+X^8(1+X+X^4)+X^{10}(1+X+X^4)+X^5(1+X^2+(X^2)^4)$$

wherein a factor like X^8 implement the offset of a voice by 8 beats, and replacement of X by X^2 renders an

$$1+X+X^2+X^3+X^4+\dots X^{14}$$

augmentation by a factor of 2; by developing, this polynomial equates to

Now one has to make a huge leap in a fantastic realm where 1+1 is no longer 2, but 0, and where the aforementioned polynomial $1+X+X^4$ has a root α (this is impossible in integers, or even real numbers). This realm exists, it is known as the Galois field with 16 elements, $\mathbf{F}_{16}$.

The astute reader may have noticed that its cardinality, 16, is just one more than the mysterious value 15, and this is indeed the key to the proof, because α^{15} is actually equal to 1. I cannot go here into details, but a thorough mathematical proof was provided which gave rise to interesting generalizations, and original theorems about the existence of similar canons and bounds for their sizes. Most notably, it enabled substantial progress on Fuglede's conjecture, a fundamental but yet unsolved statement connecting different areas of mathematics! (Amiot, 2004).

3. A HAMILTONIAN GRAPH IN BEETHOVEN'S NINTH SYMPHONY

Many believe that graph theory originates with a problem of about seven bridges in the town of Königsberg. Instead, it is quite probable that the very first graph devised by Leonhard Euler in the 1730's was the graph of tones connected by intervals of fifths and thirds, the *Tonnetz* which he published in his 1739 *Tentamen novae theoriae musicae.* In its modern versions, minor thirds intervals also appear, as can be seen in red on Fig. 3 so that for instance notes C, E and G form a triangle — a major triad. All 24 major and minor triads are present. Note that, unlike the original, the modern version of the Tonnetz is finite since in equal 12-notes temperament, 12 fifths equal 7 octaves, 3 major thirds equal an octave, and so on.

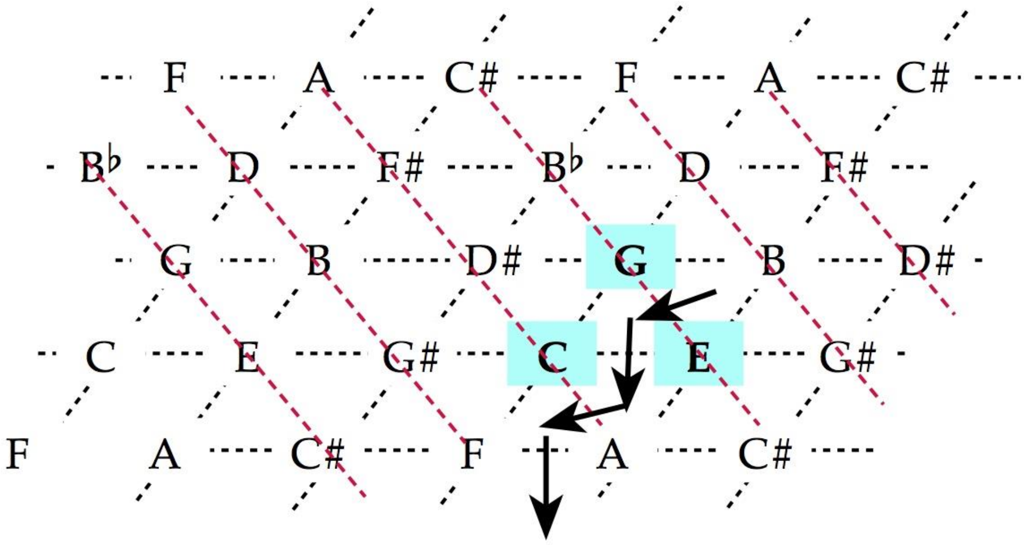

Figure 3: The modern Tonnetz

Around 1820, Beethoven wrote in the third movement of his ninth symphony a sequence of major and minor triads which are neighbours in the *Tonnetz*: moving from one to the next involved changing only one note. Such transformations were studied much later by American musicologist Richard Cohn who called them *parsimonious* (see Cohn, 1997) (a notable example is the transposition of a major scale a fifth higher, the dominant tonality). He noticed that Beethoven's sequence was a circuit, passing through each and every one of the 24 major and minor triads exactly once, in a parsimonious way. This is shown in Fig. 4.

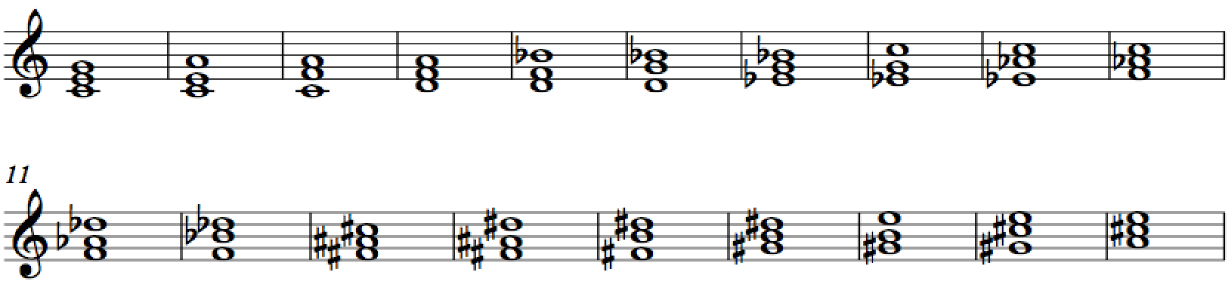

Figure 4: Beethoven's hamiltonian cycle in triads

Such circuits were introduced in graph theory 34 years after Beethoven in 1856, by Rev. William R. Hamilton, an Irish mathematician. Although Beethoven's solution appears to be quite simple (see the arrows on Fig. 3, while listening to (Beethoven, 1821)), the difficulty in finding such cycles cannot be overestimated. Indeed, it was proven in the late XXth century that looking for Hamiltonian cycles belongs to the family of *NP-complete problems*, those that the fastest conceivable computers cannot hope to solve except in very small cases. Indeed, it is only in 2009 that two Italian composers inspired by mathematics (Albini & Antonini, 2009) managed to find all 262 cycles through the 24 triangles of the Tonnetz. Several composers have used this original musical material since, for instance *Aprile* composed and played by Moreno Andreatta on a poem by Gabriele d'Annunzio (Andreatta, 2021, https://morenoandreatta.com/musicpoetry/; https://morenoandreatta.com/musicmaths/).

It is worth signalling that the story does not stop there: not only did this research (spanning from Euler to the present day) provide valuable insight into a family of graphs, but it can also be generalized to richer families of musical chords (seventh chords), though the computational complexity mentioned above has so far prevented an exhaustive enumeration of solutions.

4. PALINDROMS IN Λ

In the *Timaeus*, Plato revisited some Pythagorean mystical issues about numbers, notably the lambda-shaped diagram of powers of 2 and 3 and their multiples on Fig. 5.

Notably, at least the first numbers of the list procure frequencies for notes of the scale: if (say) 1 is an F, then all powers of 2 are octaves of this initial note, whereas 9, 18, 36… are Gs, 3, 6, 12… are Cs and so on. What happens crucially in music is a reshuffling of the powers order (see lines on Fig. 5) by the integers natural order, from bass to treble. Using the more customary combinations of octaves (2) and natural fifths (3/2) generates an infinite ascending sequence of notes:

F,C,F,G,C,D,F,G,A,C,D,E,F,G,A,B,C,D,E,F…

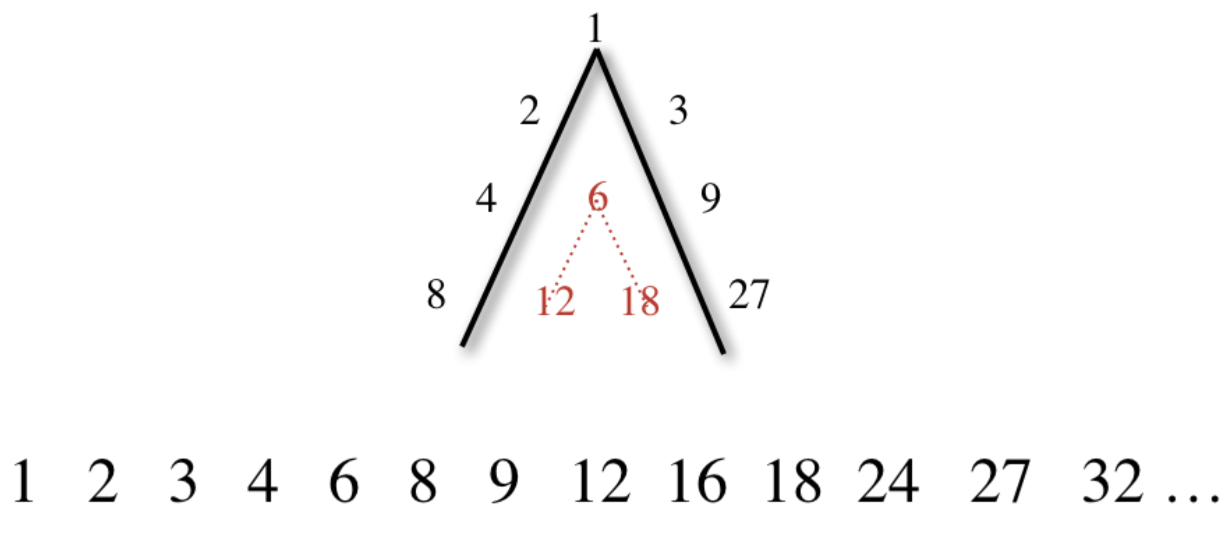

Figure 5: Plato's original Lambda from the *Timaeus*.

Norman Carey and his accomplice David Clampitt (1989) had made a thorough study of these sequences generated by two non-congruent intervals [no power of 2 equates a power of 3/2]. It was well known, for instance, that this sequence contains the pentatonic and diatonic scales, and many others besides. Furthermore, the derived sequence of the logarithms of consecutive intervals had notable features: these intervals get infinitely smaller and smaller; and when a novel interval appears, then an old one vanishes for ever, so that never more than three different intervals are active.[4] The reader may spend some time contemplating the labelled sequence of these intervals, that N. Carey called Λ after Plato:

Λ = *abcbcdccdcdedcdeddededdeddfddeddfdddfddfdddfddfgfddfddfgfddfgfdfg fdd-fgfdfgfgfdfgfdfgfgfdfgfgffgfgfdfgfgffgfgfgffgfgffgfgfgffgfgff**h**ffgfgffgfgff**h**…*

In traditional terms, *a*= ln (3/2) is a fifth, *b* =ln (4/3) a fourth, *c* =ln(9/8) a whole tone, *d* a minor third, and so on. Also notice that, for instance in the pentatonic DFGAC, the successive intervals *dccd* form a palindrome. This is well known and understood in music theory. Nonetheless, it came as a shock to Norman when he noticed (in June 2009) that palindromes are just *everywhere* in Λ. Would the reader care to have another look? To quote N. Carey in the email he sent to Clampitt and myself,

Between two adjacent occurrences of any letter, the intervening letters form a palindrom.

This is highlighted in Fig. 5 with the first two occurrences of the letter ***h*** in boldcase, framing the palindrom *ffgfgffgfgff*.

Actually, Carey proved a year later, using computational arguments unavailable to Plato (!) that the word Λ is **saturated** in palindroms, a technical notion meaning that it is impossible to have more palindroms than that (Carey, 2012).

Of course, this theorem goes far beyond the realm of perception of music, into the universe of hidden symmetries between harmonics. Word theory in the mathematical sense, including notions like complexity and palindromicity, is a thriving field, pertaining to the most abstract ideas as well as linguistics (of course), with applications to automated recognition of authorship (or forgery), or genetics, since RNA or DNA are but words written in the four-letter alphabet A, T, C, G. It is fascinating that, for instance, both genes and musical scales encode geometrical *instructions* about their own embedding in multi-dimensional spaces; or that the most famous scales exhibit special geometrical features when considered as bare words. Or that palindroms in the genome inhibit the growth of pests!

5. CONCLUSION

To me these examples show that further exploration and understanding of music necessitated better honed mathematical instruments, beyond even Xenakis's wildest and most daring dreams. Such discoveries, and even their wording, only became possible with the most recent developments of mathematical science: Galois theory on abstract fields for Johnson's canon, word theory in its most recent ventures for Carey's Lambda, graph theory and powerful computers for Beethoven-like harmonic progression. On the other hand, it can be argued that several advances in so-called hard science were made possible by musical queries. Just as it happened time and again with Physics, musical interrogations outside the field of mathematics opened new alleys of thought, helped develop new concepts, and eventually unravel original results.

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[4] Essentially this is Steiner's *three gap conjecture/theorem*. See below how *c* disappears after *e* appears.

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